ST. JOSEPH'S COLLEGE (AUTONOMOUS) BANGALORE - 560027

Term Paper on

Unique Properties about Divisibility

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CERTIFICATE

This is to certify that **Aaruni Kaushik (15EMS2701)** has satisfactorily completed the term paper on **INSERT TERM PAPER TITLE HERE** for B.Sc. course during the academic session 2015-2018 as prescribed by **ST. JOSEPH'S COLLEGE (AUTONOMOUS)**

Guide's Signature

______________.

(Mr Taral D Shah)

Date : ___________.

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Statement 1

Given the base ***n***, and a natural number ***x*** such that ***x*** = ***n***-1, and any natural number ***y*** such that ***y*** ≤ ***x***, the operation ***y***/***x*** gives the result 0.***yyyyyyyyy***...

For ***y*** > ***x***, let ***u*** = integer part of ***y***/***x*** and let ***v*** = ***y*** mod ***x***. Then, the operation ***y***/***x*** gives the result ***u***.***vvvvvvvvvvv***...

The converse of the above is also true.

Proof

Let ***n*** be the base of the number system. Let ***x*** be ***n***-1.

Case 1:
Let ***y*** be any natural number such that ***y*** ≤ ***x***.

Then,
y/***x*** = ***y***/***n*** * ***n***/***x*** = ***y***/***n*** * 1/(***x***/***n***) = ***y***/***n*** * 1/(1-(1/***n***)). -(1)

The sum to n terms of a geometric series is given by the formula Sn = ***a*** * (1 - ***r***^***n***)/(1-***r***), where ***a*** is the first term of the geometric series, ***r*** is the common ratio of the series, and ***r*** < 1.

When ***n*** tends to infinity, that is, we sum infinite terms of such a GP, the formula is reduced to Sn = ***a*** * 1/(1-***r***). - (2).

Comparing (1) and (2), we get that the expansion of ***y***/***x*** can be written in the form of summation of infinity to a geometric series with first term y/n and common ratio 1/n < 1.

That is, ***y***/***x*** = ***y***/***n*** * 1/(1-(1/***n***)) = ***y***/***n*** + ***y***/***n***^2 + ***y***/***n***^3 + ... = 0.yyyyyyyyy... . Hence, proved.

Case 2:
Let ***y*** be any natural number such that ***y***>***x***.

Let ***y***= ***ux***+***v*** (since ***y***>***x***; ***u***,***v*** are natural numbers such that ***q***<***x***).
y/***x*** = (***ux***+***v***)/***x*** = ***u***+(***v***/***x***) = ***u***+0.***vvvvvvvv***... (from case 1) = ***u***.***vvvvvvvvv***...

Hence proved.

Proof of converse is a trivial matter of following the proof for each case in reverse (start with *.***vvvvvvvv***... form, expand into sum of geometric series, use summation formula to condense into a fractional form).

Consequences

Given any number of the form *.***vvvvvvvvv***... in any given base, where * is a wildcard, can be easily transformed into its fraction form in the given base.

If ***y***=***x***, then the operation ***x***/***x*** gives the result 0.***xxxxxxxxxxx***..., from the above property. But ***x***/***x***=1. This explains why 0.***xxxxxxxx*** in base ***x***+1 is equal to 1.

Statement 2

Digit sum d of a number n satisfies $d \equiv n \bmod 9$.

The number r is the recursive digit sum of n if and only if $r \equiv n \bmod 9$

$r = 9$ if and only if 9 divides n .

Proof

Consider $N = \{1,2,3,...\}$

Consider $n \in N$, such that $n \geq 10$

Let the digit sum of n be d.

Then, d can be written as

$$d = n - 10m + [n/10] - 10m' + [n/100] - 10m'' + ...$$

where, m, m', m'', .. $\in N$

But, $m = [n/10]$, m' = [n/100], m'' = [n/1000], ...

Then,

$$d = n - 10[n/10] + [n/10] - 10[n/100] + [n/100] - ...$$

$$d = n - 9 \left([n/10] + [n/100] + [n/1000] + ... \right)$$

$$d = n - 9k, \text{ where, } k = [n/10] + [n/100] + [n/1000] + ...$$

or, $n = d + 9k$

$\Rightarrow n \equiv n \bmod 9$

if $d = 9$,

$$n = 9 + 9k$$

$$n = 9 (1 + k)$$

$\Leftrightarrow$ $n \bmod 9 = 0$

or $9 \mid n$.

if $d < 9$, then,

$d = n \bmod 9$ (since if $a < 9$, then $a = a \bmod 9$)

if $\boldsymbol{d} > 9$, then,

we can find $\boldsymbol{r} < 10$ such that

$\boldsymbol{r} \equiv \boldsymbol{d} \bmod 9$

but, $\boldsymbol{d} \equiv \boldsymbol{n} \bmod 9$

=> $\boldsymbol{r} \equiv \boldsymbol{n} \bmod 9$

but, $\boldsymbol{r} < 10$

=> $\boldsymbol{r} = \boldsymbol{n} \bmod 9$

Hence Proved

Conclusions

Digit sum $\boldsymbol{d}$ of $\boldsymbol{n}$ satisfies $\boldsymbol{d} \equiv \boldsymbol{n} \bmod 9$

Recursive digit sum $\boldsymbol{r}$ of $\boldsymbol{n}$ satisfies $\boldsymbol{r} = \boldsymbol{n} \bmod 9$

If $9 \mid \boldsymbol{n}$, then, $9 \mid \boldsymbol{d}$ and $\boldsymbol{r} = 9$.

Statement 3

Any positive integer $\boldsymbol{x}$, when repeatedly added to some arbitrary positive integer $\boldsymbol{n}$, creates a repeating pattern in the units place, with time period $\boldsymbol{t} = 10/(\boldsymbol{x},10)$

Proof

Consider $\boldsymbol{N} = \{1, 2, 3, \ldots.\}$

Consider $\boldsymbol{x} \in \boldsymbol{N}$

Consider $\boldsymbol{g} \in \boldsymbol{N}$ such that $\boldsymbol{g} = (10,\boldsymbol{x})$

Consider an arbitrary $\boldsymbol{n} \in \boldsymbol{N}$

Consider the following sequence

$$\boldsymbol{n}, \boldsymbol{n} + \boldsymbol{x}, \boldsymbol{n} + 2\boldsymbol{x}, \boldsymbol{n} + 3\boldsymbol{x}, \ldots, \boldsymbol{n} + (10/\boldsymbol{g})\boldsymbol{x}, \ldots$$

Now, $\boldsymbol{g} = (10,\boldsymbol{x})$

$\Rightarrow 10 \mid \boldsymbol{g}$ and $\boldsymbol{x} \mid \boldsymbol{g}$

then, $10/\boldsymbol{g} \in \boldsymbol{N}$ and $\boldsymbol{x}/\boldsymbol{g} \in \boldsymbol{N}$

Consider

$$\boldsymbol{n} + (10/\boldsymbol{g})\boldsymbol{x} = \boldsymbol{n} + 10\ (\boldsymbol{x}/\boldsymbol{g}) = \boldsymbol{n} + 10\boldsymbol{k}\text{ , where } \boldsymbol{k} = \boldsymbol{x}/\boldsymbol{g} \in \boldsymbol{N}.$$

$$\boldsymbol{n} + 10\ \boldsymbol{k} \equiv \boldsymbol{n} \bmod 10$$

$$\boldsymbol{n} + 10\ (\boldsymbol{x}/\boldsymbol{g}) = \boldsymbol{n} \bmod 10$$

Therefore the units digit repeats at every $10/\boldsymbol{g}$ steps.

Therefore, $\boldsymbol{t} \leq 10/\boldsymbol{g}$.

We know that units digit repeats only when adding a multiple of 10 to a number.

By definition of GCD, $10\boldsymbol{x}/\boldsymbol{g}$ is the smallest multiple of 10 achieved by $\boldsymbol{x}$.

Therefore, $10/\boldsymbol{g}$ is the least number of steps.

Therefore, $\boldsymbol{t} = 10/\boldsymbol{g}$.

Hence proved.

Glossary

1. **Base :** The number of unique digits, including zero, used to represent numbers in a positional number system.
2. **Natural Number :** The set of all positive integers ({1, 2, 3, …})
3. **Digit Sum :** The sum of the individual digits, according to their face value, in any integer.
4. **Recursive Digit Sum :** The sum of the individual digits of any integer, then the sum of the individual digits of the previous sum, and so on, till a single digit number is obtained.
5. **G.C.D :** The Greatest Common Denominator (also known as H.C.F : Highest Common Factor) is the largest integer which divides two given integer. GCD of two integers, ***a***, ***b***, is represented as (***a***,***b***).
6. **Time Period :** The shortest interval after which a periodic pattern repeats.

REFERENCES

- Elementary Number Theory by Jones and Jones
- Elementary Number Theory by Burton
- Elementary Number Theory by Niven and Zukerman